Competitive Producer Theory "How-To"

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The purpose of this document is to *review* the main steps of producer theory, starting from a production function and ending at a supply decision, in the case of a firm in a competitive industry. It may be a helpful refresher for students in classes for which 251 is a prerequisite and it may be a helpful reference for students in 251.

The Problem

Producer theory closely parallels consumer theory. In consumer theory, consumers maximized utility subject to a budget constraint, but we could have just as well had consumers minimize the cost of achieving a given level of utility and then had them maximize their utility. In producer theory, producers maximize profit, which they do by minimizing the cost of achieving a given level of output and then choosing the output level that maximizes profit subject to those costs.[1]

Producer theory starts out with a production function. Firms are assumed to maximize profit. In a competitive market, they have no influence on the price of the output; with market power they do have some influence on it. We also often assume that input markets are competitive and that inputs are not scarce, so that the prices of inputs are constant.

Since the production function represents a physical transformation process from inputs to outputs, it is reasonable to put some assumptions on the properties of that function. We assume that the production function is complete, reflexive, transitive, monotonic (because we assume we have free disposal), and convex. Some production functions are only weakly monotonic and/or weakly convex. Notice that output is cardinal, not ordinal as utility was.

Given these assumptions, we can always find a solution to the producer's problem: how much output to produce, and how to produce it (using what combination of inputs). The steps are: minimize costs of producing an arbitrary quantity of output, in which process we determine conditional factor demands; then plug conditional factor demands into the cost relationship to get cost as a function of output; then use the cost function and revenues to maximize profit. The end result of the process is the firm's supply curve: the relationship between market price and the quantity that this firm will supply.

Minimizing Costs

Think of production as a transformation from inputs or factors (which we'll designate by the variables $x_1, x_2, \ldots, x_K$ if we have *K* inputs) to an output (which we'll call *q*). Let's say that our

[1] It would be possible to do cost minimization and profit maximization together in a single big optimization problem. But it would be messy, particularly for multiple-input production, and thus we won't do it.

production function is $q = f(x_1, x_2, \ldots, x_K)$. For any given amount of output *q*, there are many different input bundles that could produce that amount of *q*. Our first goal is to find the lowest-cost input bundle that can be used to produce *q* units of output.

Costs come from buying inputs. Since we assume competitive input markets, input prices are constant—let's call them $w_1, w_2, \ldots, w_K$ respectively. So if we have only two inputs our costs are:

$$c(x_1, x_2) = w_1 x_1 + w_2 x_2$$

Optimization minimizes those costs subject to the constraint, which is the production function:

$$\min_{x_1, x_2} w_1 x_1 + w_2 x_2 \quad s.t.\, q = f(x_1, x_2)$$

This optimization process will yield conditional factor demands: $x_1(w_1, w_2, q)$ and $x_2(w_1, w_2, q)$. They are "conditional" in the sense that they condition on the level of output *q*.[2]

Now, the next step depends on whether we think we will have an interior solution (characterized by "the tangency condition") or not. Most of economics looks at tangency-type solutions, so that will be our focus, and it demonstrates the marginal conditions that are have very important intuition. However, some non-tangency cases are important—the cases of perfect substitutes and fixed proportion (perfect complement) production both demonstrate useful intuition, for example, and cases of corner solutions (i.e. choosing zero of one input) occur in interesting applications. (However, corner solutions can be hard, so we'll mostly ignore them here.)

Interior Solution Characterized by "The Tangency Condition"

The tangency condition states that at the cost-minimizing point, the isoquant must be tangent to the isocost—this is the lowest-cost way of producing on this isoquant.

For this to happen, the production function must be strictly convex and monotonic. This means we have nice curvy isoquants. For the solution to occur at a tangency point, it must also be true that isoquants are neither so steep nor so flat relative to the isocost as to preclude a tangency point within the positive quadrant. (In such a case, production is very well-behaved but is simply very extreme in favor of one input over the other (relative to the prices of the inputs) to the point where tangency is impossible so a corner solution (choosing zero of one good) will occur.)

[2] Unconditional factor demands, $x_1(w_1, w_2, p)$ and $x_2(w_1, w_2, p)$, are the optimal levels of input at the optimal level of output; you may use these if you do a profit maximization problem "all at once" rather than doing cost minimization before profit maximization. As you can see, they have output price (which determines optimal output level) rather than *q* (output level) as an argument.

Why does this situation give tangency? The intuition is that with this kind of production, the isoquant demonstrates diminishing marginal technical rate of substitution (MTRS). Thus the more of an input we use, the easier it is to swap some of that input out for another input. At the same time, the firm faces fixed input prices, and buying a unit of input x_1 has an opportunity cost: the amount of input x_2 we could be buying instead (w_1/w_2 units of x_2). If the firm is profit maximizing, it must not be the case that the firm could lower costs by making such an input swap. Thus the firm trades with the market until the rate at which they can technically swap the goods for each other in production (the MTRS) equals the rate at which the market will trade them for each other (the input price ratio).

We can prove this analytically. Let's solve the firm's cost minimization problem as a constrained optimization problem:

$$\min_{x_1,x_2} w_1x_1 + w_2x_2 \quad s.t.\, q = f\left(x_1, x_2\right)$$

We will use the Lagrangian method to demonstrate the solution.[3] We build a Lagrangian equation using λ as the Lagrange multiplier. The multiplier λ is mainly a part of the "math trick" that is the Lagrangian process here; because of the assumption that the constraint is binding, λ is a "nuisance parameter" in the sense that we won't try to identify its value and we'll just try to get rid of it in the algebra of the first order conditions.[4]

$$\mathcal{L} = w_1x_1 + w_2x_2 + \lambda\left(q - f\left(x_1, x_2\right)\right)$$

We take the first order conditions: take the first derivatives of the Lagrangian function with respect to x_1, x_2, and λ, and set each equal to zero:

FOC#1: $\dfrac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda \dfrac{\partial f\left(x_1, x_2\right)}{\partial x_1} = 0$

FOC#2: $\dfrac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda \dfrac{\partial f\left(x_1, x_2\right)}{\partial x_2} = 0$

FOC#3: $\dfrac{\partial \mathcal{L}}{\partial \lambda} = q - f\left(x_1, x_2\right) = 0$

[3] Why the Lagrangian? Those of you who have used Lagrangians in other settings may know that because we assume a binding constraint, we're not really using the full powers of the Lagrangian. Instead of the Lagrangian, we could do this problem by solving the constraint (since it's binding) for one of the variables and substituting that into the cost function. However, since the algebra of this is often messy, the Lagrangian method is often easier.

[4] The Lagrange multiplier λ actually has a meaningful interpretation, but we don't take advantage of it in this class very much. In this problem, λ is the marginal cost savings that would come from producing a tiny bit more for a given amount of inputs.

Now, remember that we want to get rid of those λ's, and solve for $x_1(w_1, w_2, q)$ and $x_2(w_1, w_2, q)$. So the next thing I always do is rearrange FOC#1 and FOC#2 to put the λ term on the right-hand side, and then I divide FOC#1 by FOC#2:

$$\frac{w_2 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_2}}{w_1 = \lambda \frac{\partial f(x_1, x_2)}{\partial x_1}}$$

Our nuisance parameter λ cancels out, and we are left with:

$$\frac{\frac{\partial f(x_1, x_2)}{\partial x_1}}{\frac{\partial f(x_1, x_2)}{\partial x_2}} = \frac{w_1}{w_2}$$

But the left-hand side is just the (negative of the) MTRS, since:

$$MTRS = -\frac{MP_1}{MP_2}$$

We just derived the tangency condition by solving the cost minimization problem, thus proving that the problem is generally solved at tangency!

If your production function will give you a solution with tangency, you can either use the Lagrangian process or you can skip the Lagrangian process and start from the condition that (minus) MTRS = input price ratio. Me, I do not skip forward because I'm afraid that I'll write either the MTRS or the input price ratio upside down if I do.

But how does that solve the problem? You have two equations (the tangency condition from your first two FOCs and the production function, which was your third FOC) and two unknowns, and you just need to solve for your unknowns $x_1(w_1, w_2, q)$ and $x_2(w_1, w_2, q)$ and you'll be done.

Start from these two equations:

$$\frac{\frac{\partial f(x_1, x_2)}{\partial x_1}}{\frac{\partial f(x_1, x_2)}{\partial x_2}} = \frac{w_1}{w_2}$$

$$q = f(x_1, x_2)$$

Then solve for the variables of interest!

Tangency Solution Example

Let's do this with a simple Cobb-Douglas production function: $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$. Again, Cobb-Douglas is not the only possible strictly convex production function—it just has nice derivatives.

Here, our Lagrangian function is:

$$\mathcal{L} = w_1 x_1 + w_2 x_2 + \lambda\left(q - x_1^{1/3} x_2^{1/3}\right)$$

This function gives us the first order conditions:

FOC#1: $\frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda\left(\frac{x_2^{1/3}}{x_1^{2/3}}\right) = 0$ … and we can rearrange to: $w_1 = \lambda\left(\frac{x_2^{1/3}}{x_1^{2/3}}\right)$

FOC#2: $\frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda\left(\frac{x_1^{1/3}}{x_2^{2/3}}\right) = 0$ … and we can rearrange to: $w_2 = \lambda\left(\frac{x_1^{1/3}}{x_2^{2/3}}\right)$

FOC#3: $\frac{\partial \mathcal{L}}{\partial x_2} = q - x_1^{1/3} x_2^{1/3} = 0$ … and we can rearrange to: $q = x_1^{1/3} x_2^{1/3}$

By dividing FOC#1/FOC#2, we get the tangency condition of:

$$\frac{w_1}{w_2} = \frac{x_2}{x_1}$$

We can then solve that for x_2…

$$x_2 = \frac{w_1}{w_2} x_1$$

We plug that into the production function…

$$q = x_1^{1/3}\left(\frac{w_1}{w_2} x_1\right)^{1/3} = \left(\frac{w_1}{w_2}\right)^{1/3} x_1^{2/3}$$

We can then solve that for our conditional factor demand function for x_1:

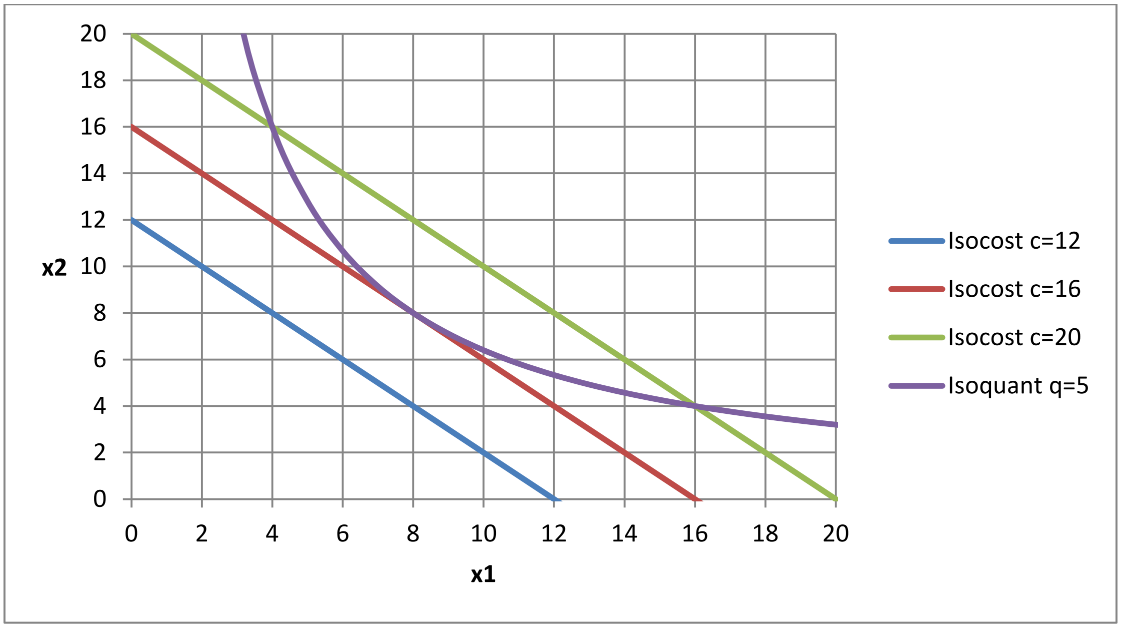
$$\boxed{x_1(w_1, w_2, q) = \left(\frac{w_2}{w_1}\right)^{1/2} q^{3/2}}$$

We plug that into our tangency condition above to get the conditional factor demand for x_2:

$$x_2\left(w_1, w_2, q\right) = \left(\frac{w_1}{w_2}\right)^{1/2} q^{3/2}$$

Voila! You know the problem is solved because the choice variables we were trying to solve for are "all alone" on the left hand side of the equation, and on the right hand side of the equation we have only parameters—not other variables.

Here's a picture of it for $q = 4$ and $w_1 = w_2 = 1$ (which gives optimal $x_1 = \left(\frac{1}{1}\right)^{1/2} * 4^{3/2} = 8$ and $x_2 = \left(\frac{1}{1}\right)^{1/2} * 4^{3/2} = 8$).



Situations Without Tangency

When will we not have tangency of the isocost and the isoquant at the firm's optimal choice? In cases of non-strictly convex or non-strictly monotonic production or production that greatly favors one or another input relative to their price ratio (corner solutions). Two places we'll see this situation a lot are the cases of perfect substitutes (non-strictly convex) and fixed proportions production (non-strictly convex and non-strictly monotonic).

In these cases, we can often come up with a solution method—a rule that will identify the lowest-cost input bundle that produces a given quantity even though we won't have tangency. Here's how I approach perfect substitutes and fixed proportions production.

Perfect Substitutes

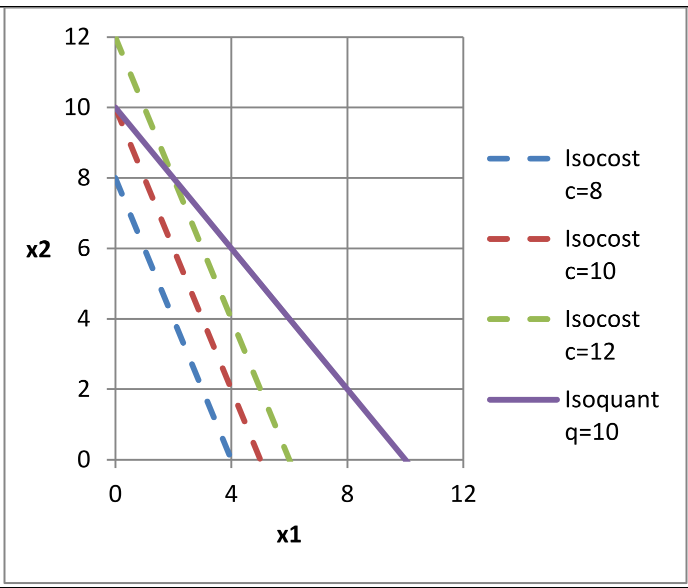
The case of perfect substitutes is a production function like this:

$$f(x_1, x_2) = x_1 + x_2$$

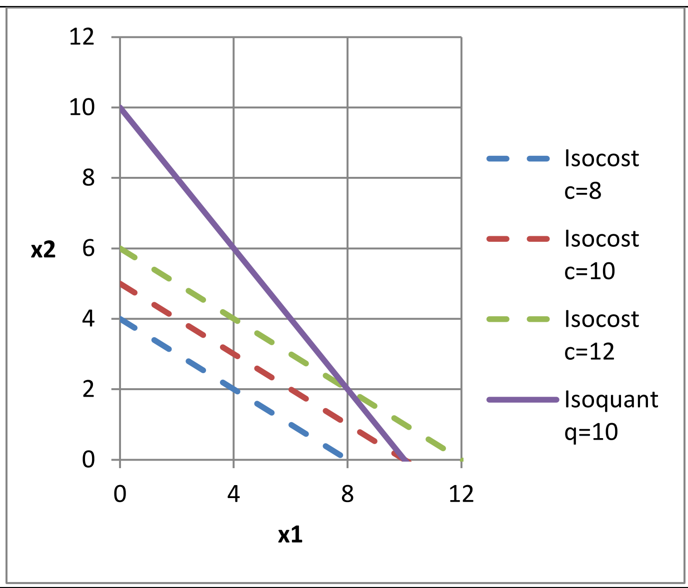
Note that when you see a plus sign you cannot necessarily infer that it's perfect substitutes. The key point is that the MTRS is constant for perfect substitutes—and this should be intuitive to you, if you think about what perfect substitutes represent. Here, the MTRS between the two inputs is 1. You may have different weights on the two goods depending on the nature of the substitutability. Perfect substitutes preclude tangency solutions because with a constant MTRS, it's impossible to trade with the market until your MTRS equals the price ratio.

The solution rule: you will use only the input that gives you more output per dollar.

We can see this graphically as well: if the isocost is steeper than the isoquant, you'll buy only the vertical-axis input; if the isocost is flatter, you'll buy only the horizontal-axis input. This is what we call a "corner solution", where one of the inputs has quantity zero. If the isocost is the same slope as the isoquant, they lie right on top of each other—and in this case, you have the same cost level with all input bundles on the isoquant. (You can also have a corner solution when production is strictly well behaved, as I've alluded to, if the isocost is just really flat or steep relative to the isoquant.)



Isocost steeper than Isoquant



Isocost flatter than Isoquant

The solution therefore looks like this:

$$x_1(w_1,w_2,q)=\begin{cases} q & if\ w_1<w_2 \\ 0 & if\ w_1>w_2 \end{cases}$$

$$x_2(w_1,w_2,q)=\begin{cases} 0 & if\ w_1<w_2 \\ q & if\ w_1>w_2 \end{cases}$$

But this is incomplete. What if $w_1 = w_2$? Then any combination of x_1 and x_2 on the isoquant will work. You can add this as a separate line to the conditional factor demand function brackets; here's one of several ways to do that:

$$\boxed{x_1(w_1,w_2,q)=\begin{cases} q & if\ w_1<w_2 \\ q-x_2 & if\ w_1=w_2 \\ 0 & if\ w_1>w_2 \end{cases}}$$

$$\boxed{x_2(w_1,w_2,q)=\begin{cases} 0 & if\ w_1<w_2 \\ q-x_1 & if\ w_1=w_2 \\ q & if\ w_1>w_2 \end{cases}}$$

Or you can change one of those < signs to ≤, which looks nicer so that's what I tend to do. Note that this looks like a bit of a cheat because it involves the assumption that if prices are equal you choose all of that one input. But in the great scheme of things it doesn't matter and it's not wrong because the input mix is arbitrary in that situation.

$$\boxed{x_1(w_1,w_2,q)=\begin{cases} q & if\ w_1\le w_2 \\ 0 & if\ w_1>w_2 \end{cases}}$$

$$\boxed{x_2(w_1,w_2,q)=\begin{cases} 0 & if\ w_1\le w_2 \\ q & if\ w_1>w_2 \end{cases}}$$

Obviously, the "if" conditions and the relationship between the input and the level of output change if the substitution ratio is different. For me, I often mess up the numbers in those "if" conditions—what I do to make sure I get them right is that I test out a few input price ratios to make sure the conditional factor demand functions make sense.

One other thing: technically, you *can* use the Lagrangian method to solve for perfect substitutes production. I don't teach you that in class because it's a bit complicated. However, it's actually more of a "full" use of a Lagrangian. You'd use two more constraints ($x_1 \ge 0$, $x_2 \ge 0$) to do it;

each gets incorporated into the Lagrangian function with their own multipliers and generally one of those constraints "binds" (either $x_1 = 0$ or $x_2 = 0$) while the other does not.

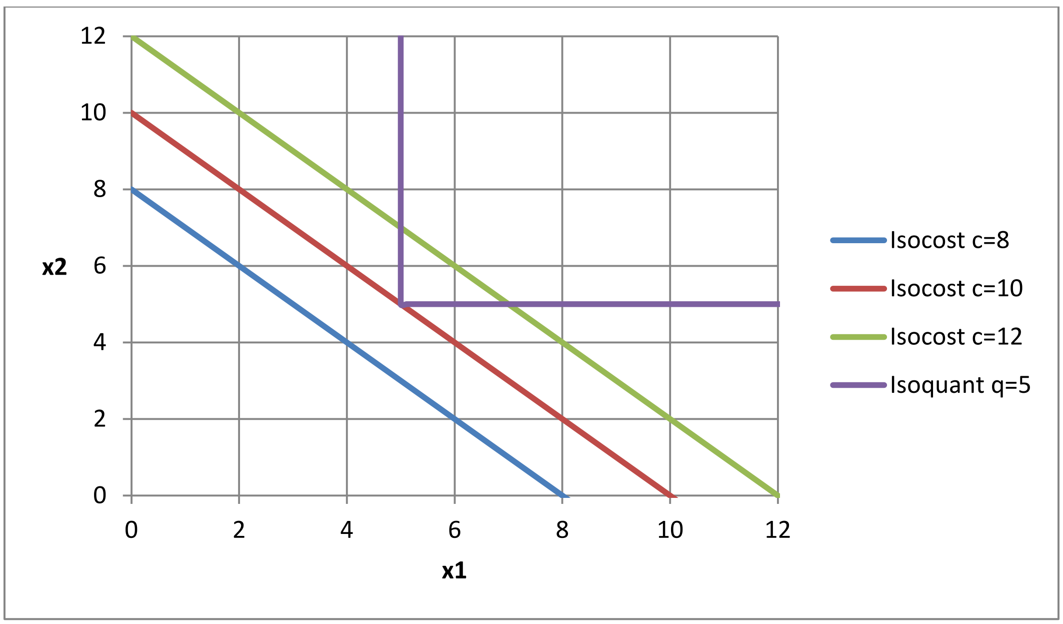
Fixed Proportions

Fixed proportions production (similar to perfect complements) has a production function like:

$$f(x_1, x_2) = \min\{x_1, x_2\}$$

You may have different weights on the inputs. The fixed ratio required by the production function above is 1:1 (e.g. hot dog and bun), but it's possible to have any other ratio (e.g. 12:1, for eggs:carton, which would be $f(x_1, x_2) = \min\{x_1, 12x_2\}$). The weights sit next to the "wrong" input; try them one way and test them out with actual numbers for the goods to see if it's right.

The solution rule is that we always want to be at the "corner" of the isoquant, i.e. we always want to use inputs in that fixed proportion. In the hot-dog-bun case, we always want $x_1 = x_2$; in the eggs-carton case we always want $x_1 = 12x_2$. Graphically, it looks like this:



How do we derive the solution algebraically? It's actually the easiest problem to solve. In this hot-dog-bun example, we are solving two equations for two unknowns:

$$x_1 = x_2$$

$$q = \min\{x_1, x_2\}$$

Clearly, if x_1 and x_2 are equal, then the output is equal to the number of each of the inputs:

$$q = x_1 = x_2$$

We can solve that out to the following conditional factor demand functions:

$$\boxed{x_1(w_1, w_2, q) = q}$$

$$\boxed{x_2(w_1, w_2, q) = q}$$

Obviously, *q* may be multiplied or divided by something if the proportion is different.

Making the Cost Function

Now that we have conditional factor demands $x_1(w_1, w_2, q)$ and $x_2(w_1, w_2, q)$, we simply plug them into our cost relationship to get our cost as a function of output.

$$c(w_1, w_2, q) = w_1 x_1(w_1, w_2, q) + w_2 x_2(w_1, w_2, q)$$

With our Cobb Douglas conditional factor demands, this is simply:

$$c(w_1, w_2, q) = w_1 \left(\frac{w_2}{w_1}\right)^{1/2} q^{3/2} + w_2 \left(\frac{w_1}{w_2}\right)^{1/2} q^{3/2} = 2 w_1^{1/2} w_2^{1/2} q^{3/2}$$

Often for brevity we write $c(q)$ instead of $c(w_1, w_2, q)$. Cost varies with the input prices, but when we look at the cost function alone we focus on the relationship between cost and output.

The Supply Curve

Now we have a cost function, so we can maximize profits without worrying about what's going on with inputs, and solely focus on level of output *q*.

$$\max_q \pi(q) = R(q) - c(q) = pq - c(q)$$

We'll have three supply conditions.

First, we know that to maximize something, we set its derivative equal to zero.

$$\frac{d\pi}{dq} = 0 \rightarrow p - MC(q) = 0 \rightarrow p = MC(q)$$

In other words, given a price set by the market, the firm produces at the quantity at which its marginal cost equals that price. Note that price is marginal revenue for a competitive firm since the firm cannot affect the price. If the firm had market power, we'd still see that marginal revenue equals marginal cost at the maximum profit condition, but marginal revenue would no longer simply be price.

Second, we know that sometimes the first order condition will give us a minimum rather than a maximum. Thus we need a second order condition. To identify a maximum as opposed to a minimum, it must be the case that the second derivative of the thing we're maximizing is negative—that the slope is declining.

$$\frac{d^2\pi}{dq^2} \leq 0 \rightarrow -c''(q) \leq 0 \rightarrow c''(q) \geq 0 \text{ or } MC'(q) > 0$$

In other words, the marginal cost must be rising, not falling. This makes sense because if MC is falling then the profit from the next unit has to be greater than the profit from the last unit, so any sane producer would expand output in that case.

These two conditions identify the maximum profit. But sometimes the maximum profit simply is not good enough. If the maximum profit puts you at a loss of $10 million per year, would you stay in business? Thus we need the no-shutdown / no-exit condition.

In the short run, a firm can't exit the industry if it's in distress. If it's losing money on production, the best it can do is shut down temporarily—send home all of its variable factors and produce zero output. If it does this, its losses are simply its fixed costs; that is, $\pi(q=0) = -F$. So in the short run, the firm compares the profit it gets at its optimized quantity with this to see whether to shut down or keep producing:

$$\pi(q^*)\ vs.\ \pi(q=0)$$

We can restate profit thus: $pq^* - c(q^*)\ vs. -F$

And then thus to make it even clearer: $pq^* - VC(q^*) - F\ vs. -F$

So those *F*'s can go away on both sides. And $AVC = \frac{VC}{q}$, so we are comparing:

$$(p - AVC)q^*\ vs.\ 0$$

Thus it's worth producing our optimal *q** if price is greater than variable cost at this level of production. We know we produce along the marginal cost curve and the marginal cost curve is everywhere above the AVC curve after the MC curve has cut through the minimum AVC. Therefore we produce according to the no-shutdown condition if price is above minimum AVC. Note that we may still produce while making negative profits here, but at least we'll be covering our variable costs and perhaps eating a bit into our fixed costs as well.

In the long run, what's different is that firms can exit the market. Thus the relevant comparison is to the rates of return we could earn in other industries, which is the market rate of return $\pi = 0$. Thus the comparison is:

$$\pi(q^*)\ vs.\ \pi = 0$$

Which we rewrite as: $pq^* - c(q^*)\ vs.\ 0$

Now note that $AC = \frac{c(q)}{q}$, so we are comparing:

$$(p - AC)q^*\ vs.\ 0$$

Thus in the long run, we exit the industry if price drops below minimum average cost. We stay in if price is above minimum average cost. Note, of course, that in the long run with free entry and exit, we expect firms to enter a market with positive profits and to leave a market with negative profits. Therefore in the long run, profits in a market like this will be driven to zero and the price will be driven to the minimum average cost.

So the supply curve is the marginal cost curve but only where the marginal cost curve is sloped upward and where it is above minimum average cost (for the long run) or minimum average variable cost (for the short run).

For our Cobb-Douglas example, we have a cost of $c(q) = 2w_1^{1/2}w_2^{1/2}q^{3/2}$. When we take the derivative, this gives us a marginal cost of $MC(q) = 3w_1^{1/2}w_2^{1/2}q^{1/2}$. In this case, if there are no fixed costs, the short run and long run supply curves will be identical to the marginal cost curve. This is because the AC and AVC curves will always be below the MC $\left(AC = AVC = \frac{c}{q} = \frac{2w_1^{1/2}w_2^{1/2}q^{3/2}}{q} = 2w_1^{1/2}w_2^{1/2}q^{1/2}\right)$, so if we produce where price is marginal cost then price is always above minimum average cost and minimum average variable cost.

If there are fixed costs, then the supply curve in the short run is the same because average variable costs are the same as marginal costs. In the long run, however, the supply curve is the marginal cost curve above the minimum average cost. Here, the cost function is $c(q) = 2w_1^{1/2}w_2^{1/2}q^{3/2} + F$ so the average costs are $AC = \frac{c}{q} = \frac{2w_1^{1/2}w_2^{1/2}q^{3/2} + F}{q} = 2w_1^{1/2}w_2^{1/2}q^{1/2} + \frac{F}{q}$. Mathematically minimizing this function is not so much fun, so whenever I ask you to do something like this I'm generally just looking for a graphical / intuitive answer rather than a precise *p* and *q* above which we will produce.