Consumer Choice Problem “How-To”

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The purpose of this document is to *review* how the standard consumer choice problem is done in neoclassical microeconomics. It may be a helpful refresher for students in classes for which 251 is a prerequisite and it may be a helpful reference for students in 251.

The Problem

The consumer choice problem is built from two primitives (basic and irreducible concepts):

- Preferences – what do you like? I.e., how do you rank bundles?
- Budget – what is affordable? I.e., what is in the feasible set?

Neoclassical economics assumes that people, being rational optimizers, choose the best bundle they can afford. That means they choose the most highly preferred bundle in their budget sets.

The budget set is pretty simple, and is defined by prices and income—though of course these need not be denominated in dollars, and can instead represent time, effort, etc. The only assumption we make about the budget is that prices are nearly always assumed to be strictly positive (excluding a price of zero).

But we do need some to make some assumptions about preferences to be able to solve the consumer choice problem. These assumptions are the axioms of consumer choice. It can be shown that if preferences are “well-behaved” (they conform to the axioms of consumer choice: complete, reflexive, transitive, and continuous) then we can represent those preferences with a utility function. A utility function, for the present purposes, is simply a mapping from bundles to numbers such that if you compare any two bundles, the more preferred bundle will have a higher utility number. This is ordinal utility, meaning that the numbers mean nothing on their own—they are just a bundle’s ranking. If preferences are also monotonic and convex (the two additional axioms we usually assume), we can be assured of a nice solution to the consumer choice problem.

Fundamentally, we want to predict and describe people’s actions. An action is a choice among possible actions—e.g. a choice of one bundle of goods or attributes from among many possible bundles. We assume that the way people choose is to pick the “best” bundle that is feasible. So the consumer choice problem comes down to: what’s the most-preferred bundle (the bundle that gives the highest utility) in my budget set?

Now, how do we solve the problem—how do we find that bundle? It depends on whether we think we will have an interior solution characterized by “the tangency condition” or not. Most of neoclassical economics looks at tangency-type solutions, so they will be our focus; they’re particularly interesting because they demonstrate the marginal conditions with clear and very important intuition. However, some non-tangency cases are important—the cases of perfect substitutes and perfect complements both demonstrate useful intuition, for example, and cases of corner solutions occur in interesting applications. (However, corner solutions are often difficult to analyze, so we'll mostly ignore those here.)

Interior Solution Characterized by "The Tangency Condition"

The tangency condition says that the indifference curve must be tangent to the budget constraint.

In this situation, preferences are not just well-behaved but also strictly convex and monotonic. (Many definitions of "well-behaved" will admit preferences that are non-strictly convex or non-strictly monotonic, e.g., perfect substitutes and perfect complements). This means we have nice curvy indifference curves.

For the solution to occur at a tangency point, it must also be true that indifference curves are neither so steep nor so flat relative to the budget constraint that they wouldn't have a tangency condition within the positive quadrant. In these very steep/flat cases, well-behaved preferences are tilted so strongly toward one good so that tangency is impossible—it is intuitively sensible that if you *would* trade chocolate for fermented tofu but only at a maximum rate of 10,000-to-1, you will probably often completely skip the stinky tofu, and this is all we mean when we talk about a corner solution (choosing zero of one good).

So ignoring the corner case situation, if preferences are well-behaved and strictly convex and monotonic, why must it be true that we have tangency? The intuition is that these preferences have a diminishing marginal rate of substitution (MRS). That means that as I consume more of something, I'm more willing to trade it off for other goods. At the same time, I'm facing fixed prices on the market. The ratio of prices I'm facing tells me the opportunity cost of the good I'm holding—if I hold a unit of x_1, I could trade it for p_1/p_2 units of x_2. Would I be happier having made that trade? At the optimal choice, it must be true that trading with the market would NOT make me happier (or else I'd keep trading and thus it must not be optimal). That is, I will trade with the market until the rate at which I'm willing to trade off between the goods (MRS) is the rate at which the market will allow me to trade off between the goods (the price ratio).

We can prove this analytically. Let's solve the consumer choice problem as a constrained optimization problem. Our goods are x_1 and x_2, and income is m.

$$\max_{x_1, x_2} u(x_1, x_2) \text{ subject to } p_1 x_1 + p_2 x_2 \le m$$

Because of the assumption of strict monotonicity, the budget constraint binds: we spend all our income because if we had money left over, we could make ourselves happier by buying more stuff. Thus $p_1 x_1 + p_2 x_2 = m$.

We will use the Lagrangian method to demonstrate the solution.[1] We build a Lagrangian equation using λ as the Lagrange multiplier. The multiplier λ is mainly a part of the "math trick" that is the Lagrangian process here; because of the assumption that the constraint is binding, λ is a "nuisance parameter" in the sense that we won't try to identify its value and we'll just try to get rid of it in the algebra of the first order conditions.[2]

$$\mathcal{L} = u\left(x_1, x_2\right) + \lambda\left(m - p_1 x_1 - p_2 x_2\right)$$

We take the first order conditions: take the first derivatives of the Lagrangian function with respect to x_1, x_2, and λ, and set each equal to zero:

FOC#1: $\dfrac{\partial \mathcal{L}}{\partial x_1} = \dfrac{\partial u\left(x_1, x_2\right)}{\partial x_1} - \lambda p_1 = 0$

FOC#2: $\dfrac{\partial \mathcal{L}}{\partial x_2} = \dfrac{\partial u\left(x_1, x_2\right)}{\partial x_2} - \lambda p_2 = 0$

FOC#3: $\dfrac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0$

Remember we want to get rid of those λ's, and solve for $x_1\left(p_1, p_2, m\right)$ and $x_2\left(p_1, p_2, m\right)$, because those are our demand functions. So the next thing I always do is rearrange FOC#1 and FOC#2 to put the λp term on the right-hand side, and then I divide FOC#1 by FOC#2:

$$\frac{\dfrac{\partial u\left(x_1, x_2\right)}{\partial x_1} = \lambda p_1}{\dfrac{\partial u\left(x_1, x_2\right)}{\partial x_2} = \lambda p_2}$$

Our nuisance parameter λ cancels out, and we are left with:

$$\frac{\dfrac{\partial u\left(x_1, x_2\right)}{\partial x_1}}{\dfrac{\partial u\left(x_1, x_2\right)}{\partial x_2}} = \frac{p_1}{p_2}$$

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[1] Why the Lagrangian? Those of you who have used Lagrangians in other settings may know that because we assume a binding constraint, we're not really using the full powers of the Lagrangian. Instead of the Lagrangian, we could do this problem by solving the constraint (since it's binding) for one of the variables and substituting that into the utility function. However, since that algebra can be messy, while the Lagrangian method is often easier.

[2] The Lagrange multiplier λ actually has a meaningful interpretation as a "shadow price," but we don't take advantage of it in this class. For example, in this problem, we'd interpret λ as the marginal utility of income.

But the left-hand side is just the (negative of the) MRS, since:

$$MRS = -\frac{MU_1}{MU_2}$$

We just derived the tangency condition by solving the consumer choice problem, thus proving that the problem is generally solved at tangency!

If you're in a situation where you think that your preferences will give you a solution with tangency, sometimes you can feel comfortable just skipping the Lagrangian process and starting from the condition that MRS = -price ratio. (I usually do not skip forward because I'm afraid that I'll write either the MRS or the price ratio upside down if I do.)

But how does that get us a solution to the problem? You have two equations (the tangency condition and the budget constraint) and two unknowns, and you just need to solve for your unknowns $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$ and you'll be done.

Start from these two equations:

$$\frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = \frac{p_1}{p_2}$$

$$p_1 x_1 + p_2 x_2 = m$$

Then solve for the variables of interest!

Tangency Solution Example

Let's do this with a simple Cobb-Douglas utility function: $u(x_1, x_2) = x_1 x_2$. Note that Cobb-Douglas is NOT the only strictly well-behaved utility function; we just use it a lot because the derivatives are easier than for many other functions.

Here, our Lagrangian function is:

$$\mathcal{L} = x_1 x_2 + \lambda(m - p_1 x_1 - p_2 x_2)$$

This function gives us the first order conditions:

FOC#1: $\frac{\partial \mathcal{L}}{\partial x_1} = x_2 - \lambda p_1 = 0$ … and we can rearrange to: $x_2 = \lambda p_1$

FOC#2: $\frac{\partial \mathcal{L}}{\partial x_2} = x_1 - \lambda p_2 = 0$ … and we can rearrange to: $x_1 = \lambda p_2$

FOC#3: $\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0$

By dividing FOC#1/FOC#2, we get the tangency condition of:

$$\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

We can then solve that for x_2…

$$x_2 = \frac{p_1}{p_2} x_1$$

We plug that into the budget constraint…

$$p_1 x_1 + p_2 \frac{p_1}{p_2} x_1 = m$$

We can then solve that for our demand function for x_1:

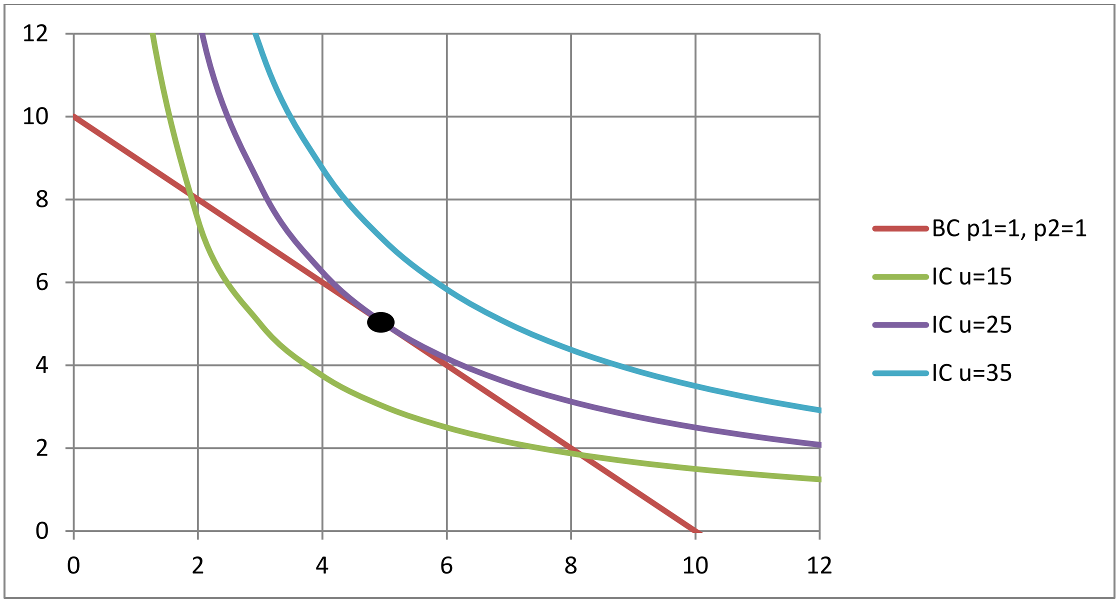
$$\boxed{x_1\left(p_1, p_2, m\right) = \frac{m}{2p_1}}$$

We can plug that into our condition above to get the demand function for x_2:

$$\boxed{x_2\left(p_1, p_2, m\right) = \frac{m}{2p_2}}$$

Voila! You know the problem is solved because the choice variables we were trying to solve for are "all alone" on the left hand side of the equation, and on the right hand side of the equation we have only parameters—not other variables.

Here's a picture of it for m = 10 and $p_1 = p_2 = 1$ (which gives optimal $x_1 = \frac{10}{2*1} = 5$ and $x_2 = \frac{10}{2*1} = 5$).



No Tangency Condition

When will we not have a tangency condition? In cases of non-strictly convex or non-strictly monotonic preferences or very extreme preferences for or against one of the goods (corner solutions). Two places we'll see this situation a lot are the cases of perfect substitutes (non-strictly convex) and perfect complements (non-strictly convex and non-strictly monotonic).

In these cases, we can often come up with a solution method—a rule that will identify the most-preferred bundle in the budget set even though we won't have tangency. Here's how I approach perfect substitutes and complements.

Perfect Substitutes

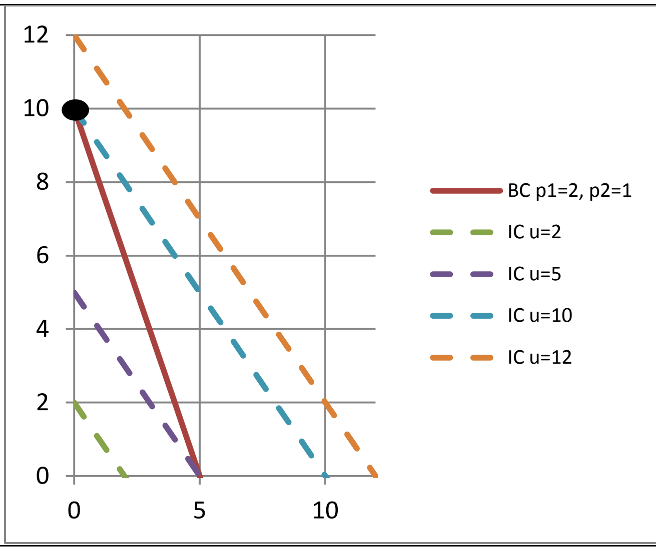
Perfect substitutes are represented by a utility function like this (where *a* and *b* are positive):

$$u(x_1, x_2) = ax_1 + bx_2$$

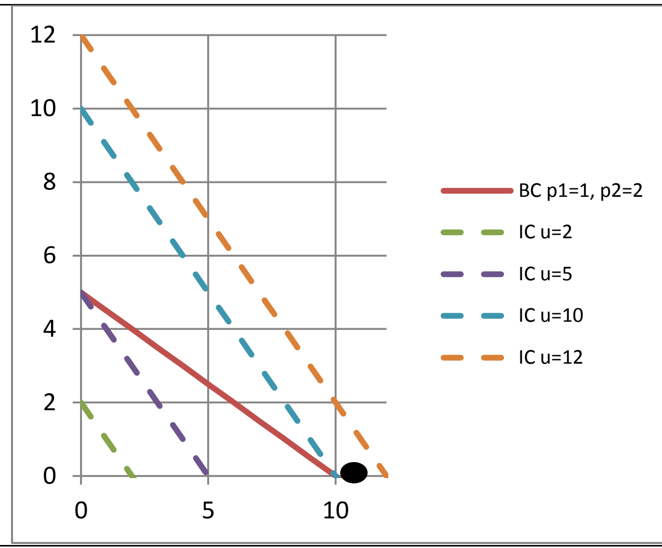
Now, just because you see a plus sign in the utility function, you can't assume you have perfect substitutes. Perfect substitutes only occur when the MRS is constant—and this should be intuitive to you if you think about what perfect substitutes represent. Here, the MRS between the two goods is a/b. You may have different weights *a* and *b* on the goods depending on the nature of the substitutability. Perfect substitutes preclude tangency solutions because with a constant MRS, it's impossible to trade with the market until your MRS equals the price ratio.

The solution rule: spend all your income on the good that gives you more utility per dollar.

We can see this graphically as well: if the budget constraint is steeper than the indifference curve, you'll buy only the vertical-axis good; if the budget constraint is flatter, you'll buy only the horizontal-axis good. This is a "corner solution", where one of the goods has quantity zero. If the budget constraint is the same slope as the indifference curve, they lie right on top of each other—and in this case, you are indifferent between all bundles on the budget constraint.



Budget Constraint steeper than Indifference Curve



Budget Constraint flatter than Indifference Curve

For tidiness, let's assume $a = b = 1$. Then the solution therefore looks like this:

$$x_1(p_1, p_2, m) = \begin{cases} \frac{m}{p_1} & if \ p_1 < p_2 \\ 0 & if \ p_1 > p_2 \end{cases}$$

$$x_2(p_1, p_2, m) = \begin{cases} 0 & if \ p_1 < p_2 \\ \frac{m}{p_2} & if \ p_1 > p_2 \end{cases}$$

But this is incomplete. What if $p_1 = p_2$ (the case where the IC lies on top of the BC)? Then any combination of x_1 and x_2 such that the budget constraint holds will work. You can add this as a separate line to the demand function brackets; here's one of several ways to do that:

$$\boxed{x_1(p_1, p_2, m) = \begin{cases} \frac{m}{p_1} & if \ p_1 < p_2 \\ \frac{m - p_2 x_2}{p_1} & if \ p_1 = p_2 \\ 0 & if \ p_1 > p_2 \end{cases}}$$

$$x_2(p_1, p_2, m) = \begin{cases} 0 & if\ p_1 < p_2 \\ \dfrac{m - p_1 x_1}{p_2} & if\ p_1 = p_2 \\ \dfrac{m}{p_2} & if\ p_1 > p_2 \end{cases}$$

But since it doesn't matter which good you choose in that one equal-price case, we typically just assume you buy all of, say, the first good. So we can change one of those < signs to ≤ and get rid of the extra line, and that cleans it up a bit.

$$x_1(p_1, p_2, m) = \begin{cases} \dfrac{m}{p_1} & if\ p_1 \le p_2 \\ 0 & if\ p_1 > p_2 \end{cases}$$

$$x_2(p_1, p_2, m) = \begin{cases} 0 & if\ p_1 \le p_2 \\ \dfrac{m}{p_2} & if\ p_1 > p_2 \end{cases}$$

Obviously, the "if" conditions change if the substitution ratio is different. For me, I often mess up the numbers in those "if" conditions—what I do to make sure I get them right is that I test out a few price ratios to make sure the demand functions make sense.

One other thing: technically, you *can* use the Lagrangian method to solve for perfect substitutes demand. I don't teach you that in class because it's much more complicated. However, it's actually more of a "full" use of a Lagrangian. You'd use two more constraints ($x_1 \ge 0$, $x_2 \ge 0$) to do it; each gets incorporated into the Lagrangian function with their own multipliers and you have some more fanciness stemming from the fact that nearly always we have the condition that one of those constraints "binds" (either $x_1 = 0$ or $x_2 = 0$) while the other does not.

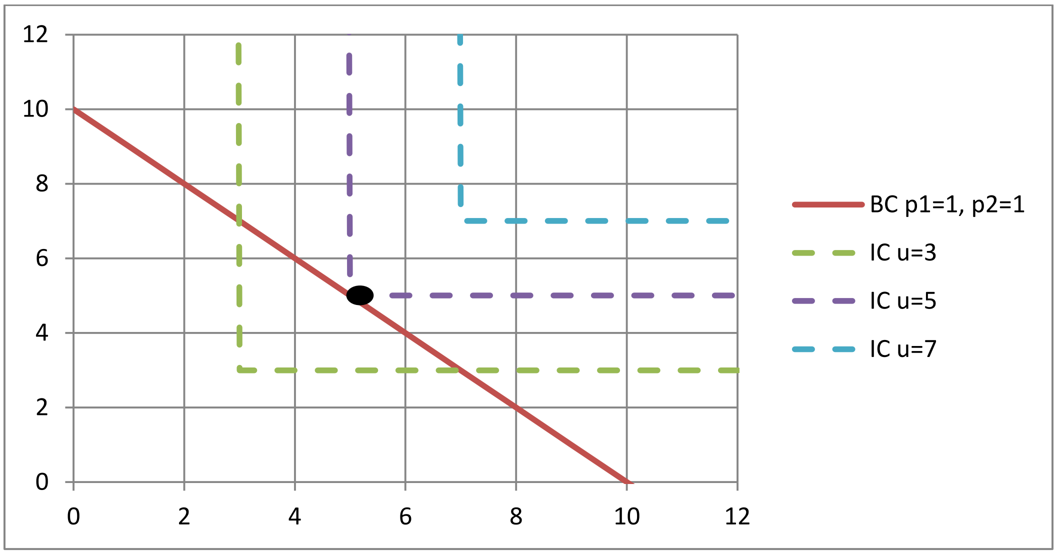
Perfect Complements

The case of perfect complements is a utility function like this:

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

You may have different weights *a* and *b* on the goods. We think of perfect complements as being a case where we consume in fixed ratio, and the fixed ratio required by the utility function above is *a*:*b*, which might be 1:1 (e.g. left shoe and right shoe), but it's possible to have any other ratio (e.g. 10:1, for coffee:cream, which would be $u(x_1, x_2) = \min\{x_1, 10x_2\}$). Notice that the coefficients seem to go onto the "wrong" good—e.g. the 10 in that example goes on cream instead of coffee. You'll get them right if you just try out the weights one way and see if they work. Here, if you have 10 of x1 (coffee), you need 1 of x2 (cream), which is right; if it was the other way you'd need 1 unit of coffee for 10 units of cream (which would be gross).

The solution rule is that we always want to be at the "corner" of the indifference curve, i.e. we always want to consume in that fixed proportion. In the left shoe-right shoe case, we always want $x_1 = x_2$; in the coffee-cream case we always want $x_1 = 10x_2$. Graphically, it looks like this:



How do we derive the solution algebraically? It's actually the easiest problem to solve, which is one reason profs use perfect complements so often. In this left shoe-right shoe example, we are solving two equations for two unknowns:

$$x_1 = x_2$$

$$p_1 x_1 + p_2 x_2 = m$$

We can solve that out to the following demand functions:

$$\boxed{x_1 = \frac{m}{p_1 + p_2}}$$

$$\boxed{x_2 = \frac{m}{p_1 + p_2}}$$

Obviously, the denominator is different if the proportion is different.